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Three-Mode Nonlinear Bogoliubov Transformations

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Abstract We introduce the three-mode nonlinear Bogoliubov transformations based on the work of Siena et al. (Phys. Rev. A 64:063803, 2001) and Ying Wu (Phys. Rev. A 66:025801, 2002) about nonlinear Bogoliubov transformations. We show that three-mode nonlinear Bogoliubov transformations can be constructed by the combination of two unitary transformations, a coordinate-dependent displacement followed by the standard squeezed transformation. Such decomposition turns all the nonlinear canonic coordinate-dependent Bogoliubov transformations into essentially linear problems as we shall prove and hence greatly facilitate calculations of the properties and the quantities related to the nonlinear transformations.

Keywords Coordinate-dependent three-mode nonlinear Bogoliubov transformations · Three-mode squeezed states

1 Introduction

In recent years Bogoliubov transformation and its various generalizations are a powerful tool in dealing with many problems in quantum systems in various fields [4–12]. However, the transformations involved in the most of references are linear in nature and hence can only deal with either the model in the bilinear form of the annihilation and creation operator, or in a special kind of nonlinear transformations such as the angular momentum representation of the bosonic operators. But many quantum systems involve nonlinear interaction and hence it is desirable to consider nonlinear Bogoliubov transformations. Siena et al. and Ying Wu have achieved some progress in this direction. They have shown that some coordinate-dependent Bogoliubov transformations are indeed canonical transformations when some of their transformation parameters satisfy certain requirement and have discussed the corresponding multiphoton states.

The most general coordinate-dependent Bogoliubov transformations mentioned by Siena et al., which have the following form $b = \mu a + \nu a^\dagger + \gamma F(X_1) + \eta G(X_2)$, $b^\dagger = \mu^* a^\dagger + \nu^* a +$

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$\gamma^* F(X_1)+\eta^* G(X_2)$, where $a, a^\dagger$ satisfy the commutative relation $[a, a^\dagger]=1$, μ, ν, γ, η are complex parameters, and $F=F^\dagger$ and $G=G^\dagger$ are operator-valued nonlinear functions of the fundamental coordinate operators: $X_1=\frac{a+a^\dagger}{\sqrt{2}}$, $X_2=\frac{a-a^\dagger}{\sqrt{2}i}$, which obviously satisfy the commutative relation $[X_1, X_2]=\frac{i}{2}$. When some of their transformation parameters satisfy certain requirements, they have shown that some coordinate-dependent Bogoliubov transformations are indeed canonical transformations. Based on the work of Siena et al., Ying Wu have shown that the coordinate-dependent Bogoliubov transformations can be constructed by the combination of two unitary transformations, a coordinate-dependent displacement followed by the standard squeezed transformation. Such a decomposition make nonlinear problems reduced to linear ones and the calculations have been transformed into those involving only the standard squeezed states.

Thus a question naturally arises: under which conditions the nonlinear three-mode coordinate-dependent Bogoliubov transformations are canonical transformations? If all the corresponding canonical ones can be constructed by the combination of two unitary transformations, a coordinate-dependent displacement followed by the standard squeezed transformation? Because three-mode nonlinear Bogoliubov transformations is more complicated than the single mode, to answer these question we must first use the usual three-mode squeezing transformation and then analyze its nonlinear Bogoliubov transformations.

2 Coordinate- and Momentum-Dependent Three-Mode Squeezing Transformations

Let us consider a three-mode bosonic quantum system. The usual three-mode squeezing transformation is [3]

$$S_3 a_i S_3^{-1} = a_i \cosh\lambda + A_{ij} a^\dagger \sinh\lambda, \tag{1}$$

where $i=1,2,3$, $A_{ij}=\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$, $a^\dagger=\begin{bmatrix} a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \end{bmatrix}$, $S_3=\exp[-\lambda(A^\dagger - A)]$, $A^\dagger = \frac{1}{6}\sum_{i=1}^3 a_i^{\dagger 2} - \frac{2}{3}\sum_{i<j} a_i^\dagger a_j^\dagger$, $A=\frac{1}{6}\sum_{i=1}^3 a_i^2 - \frac{2}{3}\sum_{i<j} a_i a_j$.

We now introduce the following squeezing transformations:

$$b_i = a_i \cosh\lambda + A_{ij} a^\dagger \sinh\lambda + i\pi_i F_i + \pi_{i+3} G_i, \tag{2}$$

where $\pi=\tau, k, l$, $\pi_i=\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}$, $\pi_{i+3}=\begin{bmatrix} \pi_4 \\ \pi_5 \\ \pi_6 \end{bmatrix}$, $F_i=[\,F_1\ F_2\ F_3\,]$, $G_i=[\,G_1\ G_2\ G_3\,]$, and $F_i=F_i^\dagger$ $(i=1,2,3)$ are arbitrary nonzero operator-valued function of the coordinate operators X_i $(i=1,2,3)$, $G_i=G_i^\dagger$ $(i=1,2,3)$ are arbitrary nonzero operator operator-valued functions of the momentum operators P_i $(i=1,2,3)$.

Let us first discuss the conditions under which conditions the operators b_i and $b_i^\dagger$ $(i=1,2,3)$ satisfy the canonical commutation relations $[b_i, b_j^\dagger]=\delta_{ij}$, $[b_i, b_j]=0$ $(i=1,2,3)$. By means of the operator formulas

$$\begin{aligned} &[a_i, F_i] = \frac{1}{\sqrt{2}}\frac{dF_i}{dX_i}, \qquad [a_i^+, F_i] = -\frac{1}{\sqrt{2}}\frac{dF_i}{dXi}, \\ &[a_i, G_i] = \frac{i}{\sqrt{2}}\frac{dG_i}{dP_i}, \qquad [a_i^+, G_i] = \frac{i}{\sqrt{2}}\frac{dG_i}{dP_i}, \end{aligned} \tag{3}$$

we obtain

$$
\begin{aligned}
[b_i, b_i^\dagger] = {} & 1 + \frac{i}{\sqrt{2}} \frac{dF_1}{dX_1} \left(-\frac{1}{3} \sinh\lambda + \cosh\lambda\right)(\pi_i - \pi_i^*) + \frac{\sqrt{2}i}{3} \sum_{l \neq i} \frac{dF_l}{dX_l} \sinh\lambda (\pi_l - \pi_l^*) \\
& - \frac{i}{\sqrt{2}} \frac{dG_1}{dP_1} \left(-\frac{1}{3} \sinh\lambda + \cosh\lambda\right)(\pi_{i+3} - \pi_{i+3}^*) \\
& - \frac{\sqrt{2}i}{3} \sum_{l \neq i} \frac{dG_l}{dP_l} \sinh\lambda (\pi_{l+3} - \pi_{l+3}^*) + \sum_{n=1}^{3} i (\pi_n \pi_{n+3}^* + \pi_n^* \pi_{n+3}) [F_n, G_n],
\end{aligned} \tag{4}
$$

where $\pi = \tau, k, l$, $i = 1, 2, 3$, $l = 1, 2, 3$, $i \neq l$, $n = 1, 2, 3$ and

$$
\begin{aligned}
[b_i, b_l] = {} & \frac{i}{\sqrt{2}} \left[\pi_i \left(\cosh\lambda - \frac{1}{3}\sinh\lambda\right) + \frac{2}{3} K_i \sinh\lambda\right] \frac{dF_i}{dX_i} \\
& + \frac{i}{\sqrt{2}} \left[\frac{2}{3} K_l \sinh\lambda + \pi_l \left(\cosh\lambda - \frac{1}{3}\sinh\lambda\right)\right] \frac{dF_2}{dX_2} \\
& + \frac{\sqrt{2}}{3} i (\pi_m - K_m) \sinh\lambda \frac{dF_m}{dX_m} + \frac{i}{\sqrt{2}} \left[\pi_{i+3} \left(\cosh\lambda + \frac{1}{3}\sinh\lambda\right)\right. \\
& \left. + \frac{2}{3} K_{i+3} \sinh\lambda\right] \frac{dG_i}{dP_i} - \frac{i}{\sqrt{2}} \left[\frac{2}{3} K_{l+3} \sinh\lambda + \pi_{l+3} \left(\cosh\lambda + \frac{1}{3}\sinh\lambda\right)\right] \frac{dG_l}{dP_l} \\
& + \frac{\sqrt{2}}{3} i (\pi_{m+3} - K_{m+3}) \sinh\lambda \frac{dG_m}{dP_m} + \sum_{n=1}^{3} (i\pi_n K_{n+3} - i K_n \pi_{n+3}) [F_n, G_n],
\end{aligned} \tag{5}
$$

where $i \neq l$, $i, l = 1, 2, 3$, $m \neq i$, $m \neq l$, $m = 1, 2, 3$, $\pi \neq \mathrm{K}$, $\pi, K = \tau, k, l$, $n = 1, 2, 3$. We can easy to see that for arbitrary nonzero functions, $F_i = F_i^\dagger$, $G_i = G_i^\dagger$ $(i = 1 \to 6)$ and the canonical commutation relations $[b_i, b_j^+] = \delta_{ij}$, $[b_i, b_j] = 0$ $(i = 1, 2, 3)$ lead to the results that τ_i, k_i and l_i $(i = 1 \to 6)$ are all real numbers, and the relations

(1) $k_1 = l_1, k_1 = -\frac{\tau_1(3\cosh\lambda - \sinh\lambda)}{2\sinh\lambda}$, $\tau_2 = l_2, \tau_2 = -\frac{k_2(3\cosh\lambda - \sinh\lambda)}{2\sinh\lambda}$,
$\tau_3 = k_3, l_3 = -\frac{\tau_3(\cosh\lambda - \sinh\lambda)}{2\sinh\lambda}$, $\tau_i = k_i = l_i = 0$, $(i = 4, 5, 6)$, or

(2) $k_4 = l_4, \tau_4 = \frac{-k_4(3\cosh\lambda + \sinh\lambda)}{2\sinh\lambda}$, $\tau_5 = l_5, k_5 = \frac{-\tau_5(3\cosh\lambda + \sinh\lambda)}{2\sinh\lambda}$,
$\tau_6 = \kappa_6, \tau_6 = \frac{-l_6(3\cosh\lambda + \sinh\lambda)}{2\sinh\lambda}$, $\tau_i = k_i = l_i = 0$, $(i = 1, 2, 3)$.

From above we can see that the general transformations given in (4) are not canonical transformations at all if they concludes both coordinate-dependent and momentum-dependent items.

Now let us first discuss the coordinate-dependent three-mode squeezing transformations

$$
b_i = a_i \cosh\lambda + A_{ij} a^\dagger \sinh\lambda + i\pi_i F_i, \tag{6}
$$

where τ_i, k_i and l_i $(i = 1, 2, 3)$ are all real numbers, and satisfy the relations

$$
\begin{aligned}
& k_1 = l_1, \qquad k_1 = -\frac{\tau_1(3\cosh\lambda - \sinh\lambda)}{2\sinh\lambda}, \qquad \tau_2 = l_2, \qquad \tau_2 = -\frac{k_2(3\cosh\lambda - \sinh\lambda)}{2\sinh\lambda}, \\
& \tau_3 = k_3, \qquad l_3 = -\frac{\tau_3(\cosh\lambda - \sinh\lambda)}{2\sinh\lambda}, \qquad \tau_i = k_i = l_i = 0 \quad (i = 4, 5, 6).
\end{aligned}
$$

We introduce two sets of operators $d_i, d_i^\dagger$ and $b_i, b_i^\dagger$ $(i=1,2,3)$, it has the form

$$d_1 = a_1 + \frac{3i\tau_1 F_1}{3\cosh\lambda - \sinh\lambda}, \qquad d_2 = a_2 + \frac{3i\tau_2 F_2}{2\sinh\lambda}, \qquad d_3 = a_3 + \frac{3i\tau_3 F_3}{2\sinh\lambda}, \tag{7}$$

$$b_i = S_t d_i S_t^{-1} = d_i \cosh\lambda + A_{ij} d^\dagger \sinh\lambda, \tag{8}$$

where $S_t = \exp[-\lambda(A^\dagger - A)]$, $A^\dagger = \frac{1}{6}\sum_{i=1}^3 d_i^{\dagger 2} - \frac{2}{3}\sum_{i<j} d_i^\dagger d_j^\dagger$, $A = \frac{1}{6}\sum_{i=1}^3 d_i^2 - \frac{2}{3}\sum_{i<j} d_i d_j$.

By means of (9) and (10), we can prove it satisfy the canonical commutation relations

$$[d_i, d_j^\dagger] = \delta_{ij}, \quad [d_i, d_j] = 0 \quad (i,j=1,2,3). \tag{9}$$

Obviously (10) is nothing but the standard squeezing transformation relating $d_i, d_i^\dagger$ and $b_i, b_i^\dagger$ $(i=1,2,3)$. It is then straightforward to obtain

$$b_i = S_t d_i S_t^\dagger \quad (i=1,2,3). \tag{10}$$

In addition, (9) can be rewritten as

$$d_i = U(X_1, X_2, X_3) a_i U^\dagger(X_1, X_2, X_3) \quad (i=1,2,3), \tag{11}$$

where the unitary operator U has the form

$$U(X_1,X_2,X_3) = \exp\bigg[-\frac{3\sqrt{2}i\tau_1}{3\cosh\lambda - \sinh\lambda}\int_0^{X_1} F_1(x)dx - \frac{3\sqrt{2}i\tau_2}{2\sinh\lambda}\int_0^{X_2} F_2(x)dx - \frac{3\sqrt{2}i\tau_3}{2\sinh\lambda}\int_0^{X_3} F_3(x)dx\bigg]. \tag{12}$$

Obviously, the unitary operator $U(X_1, X_2, X_3)$ describes a coordinate-dependent displacement explicitly given by (10). It is then straightforward to combine unitary transformations (1) and (10) to obtain

$$b_i = Q a_i Q^\dagger = a_i \cosh\lambda + A_{ij} a^\dagger \sinh\lambda + i\pi_i F_i, \tag{13}$$

where

$$Q = S_t U(X_1, X_2, X_3) = U(X_1, X_2, X_3) S_3. \tag{14}$$

In writing (16), we have made use of the relations (10). Equation (16) tells us that the combination unitary transformations Q of the two unitary transformations described by the two unitary operator $U(X_1, X_2, X_3)$ and S_3, respectively, indeed has the form of the coordinate-dependent squeezing transformations determined by (8).

Let us consider three-mode multiphoton states $|\psi\rangle_\beta$, which is defined as the eigenstates of the annihilation operator b, i.e., $b|\psi\rangle_\beta = \beta|\psi\rangle_\beta$, $|\psi\rangle_\beta = Q|000\rangle = U(X_1,X_2,X_3)|\lambda\rangle = U(X_1,X_2,X_3)S_3|000\rangle|\psi\rangle_\beta$ can be expressed in the coordinate representation as $\psi_\beta(x_1,x_2,x_3) = \langle x_1,x_2,x_3|\psi\rangle_\beta$, where $|x_i\rangle$ is the eigenstate of coordinate operator X_i with eigenvalue x_i, i.e., $X_i|x_i\rangle = x_i|x_i\rangle$ $(i=1,2,3)$. Noting that $\langle x_1,x_2,x_3|U(X_1,X_2,X_3) = U(x_1,x_2,x_3)\langle x_1,x_2,x_3|$, we can obtain $\psi_\beta(x_1,x_2,x_3) = \langle x_1,x_2,x_3|\psi\rangle_\beta = U(x_1,x_2,x_3)\langle x_1,x_2,x_3|\lambda\rangle$. To have the explicit expression of $\psi_\beta(x_1,x_2,x_3)$, we only need to calculate the quantity $\langle x_1,x_2,x_3|\lambda\rangle$, which is independent of the operator-valued functions

$F_1(X_1)$, $F_2(X_2)$ and $F_3(X_3)$. That is to say, all we need to do is calculate the wave function of the standard three-mode squeezed vacuum state $|\lambda\rangle$ in the coordinate representation.

Using the standard three-mode squeezed vacuum

$$|\lambda\rangle = S_3|000\rangle = \sec h^{\frac{3}{2}} \exp[-A^\dagger \tanh\lambda]|000\rangle, \tag{15}$$

where $A^\dagger = \frac{1}{6}\sum_{i=1}^{3} a_i^{\dagger 2} - \frac{2}{3}\sum_{i<j} a_i^\dagger a_j^\dagger$, we have

$$a_i S_3^\dagger|\lambda\rangle = 0, \quad C_i|\lambda\rangle = 0 \quad (i = 1, 2, 3),$$

where the operators C_i is given by

$$C_i = S_3 a_i S_3^\dagger = a_i \cosh\lambda + A_{ij} a^\dagger \sinh\lambda. \tag{16}$$

Therefore, using the results

$$\begin{aligned} a_i &= \frac{1}{\sqrt{2}}(X_i + iP_i) \to \frac{1}{\sqrt{2}}\left(x_i + \frac{\partial}{\partial x_i}\right), \\ a_i^\dagger &= \frac{1}{\sqrt{2}}(X_i - iP_i) \to \frac{1}{\sqrt{2}}\left(x_i - \frac{\partial}{\partial x_i}\right) \quad (i = 1, 2, 3), \end{aligned} \tag{17}$$

we can put (18) into the form

$$\begin{aligned} &\left\{\left(x_1 + \frac{\partial}{\partial x_1}\right)\cosh\lambda + \left[\frac{1}{3}\left(x_1 - \frac{\partial}{\partial x_1}\right) - \frac{2}{3}\left(x_2 - \frac{\partial}{\partial x_2}\right)\right.\right. \\ &\quad \left.\left. - \frac{2}{3}\left(x_3 - \frac{\partial}{\partial x_3}\right)\right]\sinh\lambda\right\}\langle x_1, x_2, x_3|\lambda\rangle = 0, \\ &\left\{\left(x_2 + \frac{\partial}{\partial x_2}\right)\cosh\lambda + \left[\frac{1}{3}\left(x_2 - \frac{\partial}{\partial x_2}\right) - \frac{2}{3}\left(x_1 - \frac{\partial}{\partial x_1}\right)\right.\right. \\ &\quad \left.\left. - \frac{2}{3}\left(x_3 - \frac{\partial}{\partial x_3}\right)\right]\sinh\lambda\right\}\langle x_1, x_2, x_3|\lambda\rangle = 0, \\ &\left\{\left(x_3 + \frac{\partial}{\partial x_3}\right)\cosh\lambda + \left[\frac{1}{3}\left(x_3 - \frac{\partial}{\partial x_3}\right) - \frac{2}{3}\left(x_1 - \frac{\partial}{\partial x_1}\right)\right.\right. \\ &\quad \left.\left. - \frac{2}{3}\left(x_2 - \frac{\partial}{\partial x_2}\right)\right]\sinh\lambda\right\}\langle x_1, x_2, x_3|\lambda\rangle = 0. \end{aligned} \tag{18}$$

The solution of (20) is

$$\begin{aligned} \langle x_1, x_2, x_3|\lambda\rangle = \frac{1}{\sqrt{2\pi}} \exp\bigg[&-\frac{x_1^2 + x_2^2 + x_3^2}{2}\cosh 2\lambda \\ &- \frac{1}{6}\sinh 2\lambda(x_1^2 + x_2^2 + x_3^2 - 4x_1x_2 - 4x_1x_3 - 4x_2x_3)\bigg]. \end{aligned} \tag{19}$$

Using the expression $U(X_1, X_2, X_3)$ given in (14), we have the explicit analytical expression of the three-mode multiphoton states $\psi_\beta(x_1, x_2, x_3)$ in the coordinate representation as follow:

$$\begin{aligned}\psi_\beta(x_1,x_2,x_3) &= U(x_1,x_2,x_3)\langle x_1,x_2,x_3|\lambda\rangle \\ &= \frac{1}{\sqrt{2\pi}}U(x_1,x_2,x_3)\exp\left[-\frac{x_1^2+x_2^2+x_3^2}{2}\cosh 2\lambda \right.\\ &\quad \left.-\frac{1}{6}\sinh 2\lambda(x_1^2+x_2^2+x_3^2-4x_1x_2-4x_1x_3-4x_2x_3)\right]. \end{aligned} \tag{20}$$

Similarly, the momentum-dependent three-mode squeezing transformations can also be constructed by the combination of two unitary transformations, a momentum-dependent displacement followed by the standard squeezing transformations, i.e.,

$$Q = U(P_1,P_2,P_3)S_3, \tag{21}$$

$$U(P_1,P_2,P_3) = \exp\left[-\frac{3\sqrt{2}i\tau_4}{3\cosh\lambda+\sinh\lambda}+\frac{3\sqrt{2i}\tau_5}{2\sinh\lambda}+\frac{3\sqrt{2i}\tau_6}{2\sinh\lambda}\right]. \tag{22}$$

Another squeezing states can be obtained by making the unitary operator Q' on the three-mode vacuum state, i.e.,

$$|\psi\rangle_{\beta'} = Q'|000\rangle = U'(P_1,P_2,P_3)|\lambda\rangle = U'(P_1,P_2,P_3)S_3|000\rangle, \tag{23}$$

we have the explicit analytical expression of the three-mode multiphoton states $|\psi\rangle_{\beta'}$ in the momentum representation as follow:

$$\begin{aligned}\psi_{\beta'}(P_1,P_2,P_3) &= U'(P_1,P_2,P_3)\langle P_1,P_2,P_3|\lambda\rangle \\ &= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{p_1^2+p_2^2+p_3^2}{2}\cosh 2\lambda \right.\\ &\quad \left.-\frac{1}{6}\sinh 2\lambda(p_1^2+p_2^2+p_3^2+4p_1p_2+4p_1p_3+4p_2p_3)\right]. \end{aligned} \tag{24}$$

3 Properties of Squeezing State $|\psi\rangle_\beta$

Now let us prove that all the expectation quantities averaged over the multiphoton states $|\psi\rangle_\beta$, i.e., $\langle A\rangle =_\beta \langle\psi|A|\psi\rangle_\beta$, can be expressed in terms the expectation values averaged over the standard three-mode squeezed vacuum $|\lambda\rangle$, i.e., $\langle A\rangle_L = \langle\lambda|A|\lambda\rangle$. The subscript "$L$" means that the averaging operation relates merely to the standard three-mode squeezed vacuum involving only linear squeezing transformations. Form (14), it is easy to obtain the basic formula for such a transition as follows:

$$\langle A(X_1,X_2,X_3,P_1,P_2,P_3)\rangle = \langle U^\dagger(X_1,X_2,X_3)AU(X_1,X_2,X_3)\rangle_L, \tag{25}$$

where A is an arbitrary function. It implies $\langle f(X_1,X_2,X_3)\rangle = \langle f(X_1,X_2,X_3)\rangle_L$, which demonstrates that the coordinate variables X_1, X_2, X_3 and their arbitrary functions are not affected by the nonlinear terms in the coordinate-dependent squeezing transformations (4). Using the commutative relations $[X_i,P_j]=i\delta_{ij}$ $(i,j=1,2,3)$, we have

$$[U(X_1,X_2,X_3),P_1] = \frac{3\sqrt{2}\tau_1}{3\cosh\lambda-\sinh\lambda}U(X_1,X_2,X_3)F_1(X_1), \tag{26}$$

$$[U(X_1, X_2, X_3), P_2] = \frac{3\sqrt{2}\tau_2}{2\sinh\lambda} U(X_1, X_2, X_3) F_2(X_2), \tag{27}$$

$$[U(X_1, X_2, X_3), P_3] = \frac{3\sqrt{2}\tau_3}{2\sinh\lambda} U(X_1, X_2, X_3) F_3(X_3). \tag{28}$$

Using

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots, \tag{29}$$

we can have

$$U^{-1}(X_1, X_2, X_3) P_1^n U(X_1, X_2, X_3) = \left[P_1 - \frac{3\sqrt{2}\tau_1}{3\cosh\lambda - \sinh\lambda} F_1(X_1) \right]^n, \tag{30}$$

$$U^{-1}(X_1, X_2, X_3) P_2^n U(X_1, X_2, X_3) = \left[P_2 - \frac{3\sqrt{2}\tau_2}{2\sinh\lambda} F_2(X_2) \right]^n, \tag{31}$$

$$U^{-1}(X_1, X_2, X_3) P_3^n U(X_1, X_2, X_3) = \left[P_3 - \frac{3\sqrt{2}\tau_3}{2\sinh\lambda} F_3(X_3) \right]^n, \tag{32}$$

and hence

$$\begin{aligned} &\langle f(X_1, X_2, X_3) g(P_1, P_2, P_3) \rangle \\ &= \left\langle f(X_1, X_2, X_3) g\left(P_1 - \frac{3\sqrt{2}\tau_1}{3\cosh\lambda - \sinh\lambda} F_1, P_2 - \frac{3\sqrt{2}\tau_2}{2\sinh\lambda} F_2, P_3 - \frac{3\sqrt{2}\tau_3}{2\sinh\lambda} F_3 \right) \right\rangle_L . \end{aligned} \tag{33}$$

Equation (35) completely describes reducing the averaging operation over the multiphoton states $|\psi\rangle_1$ to the operation over the standard three-mode squeezed vacuum $|\lambda\rangle$.

Now, let us discuss the fundamental coordinate operators. For three-mode optical field, the fundamental quadrature are defined as

$$U_1 = \frac{1}{\sqrt{6}}(X_1 + X_2 + X_3), X_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger) \quad (i = 1, 2, 3), \tag{34}$$

$$U_2 = \frac{1}{\sqrt{6}}(P_1 + P_2 + P_3), P_i = \frac{1}{\sqrt{2}i}(a_i - a_i^\dagger) \quad (i = 1, 2, 3), \tag{35}$$

It is easy to prove that U_1 and U_2 satisfy the commutative relation

$$[U_1, U_2] = \frac{i}{2}, \tag{36}$$

and uncertainty relation

$$(\Delta U_1)^2 (\Delta U_2)^2 \geq \frac{1}{16}, \tag{37}$$

where

$$(\Delta U_i)^2 = \langle U_i^2 \rangle - \langle U_i \rangle^2 \quad (i = 1, 2). \tag{38}$$

If $(\Delta U_i)^2 < \frac{1}{4}$ $(i = 1, 2)$, we can say the quadrature of the three-mode optical field is squeezed.

Form (27) and (40), we have

$$(\Delta U_1)^2 = \frac{1}{4}\exp(2\lambda), \tag{39}$$

$$(\Delta U_2)^2 = \frac{1}{4}\exp(-2\lambda) + \frac{1}{4}(\Delta F)^2, \tag{40}$$

where

$$\begin{aligned} F = {} & \frac{3\sqrt{2}\tau_1}{3\mu - \nu} F_1\left[\mu X_1 + \nu\left(\frac{1}{3}X_1 - \frac{2}{3}X_2 - \frac{2}{3}X_3\right)\right] \\ & + \frac{\sqrt{2}\tau_2}{2\nu} F_2\left[\mu X_2 + \nu\left(\frac{1}{3}X_2 - \frac{2}{3}X_1 - \frac{2}{3}X_3\right)\right] \\ & + \frac{\sqrt{2}\tau_3}{2\nu} F_3\left[\mu X_3 + \nu\left(\frac{1}{3}X_3 - \frac{2}{3}X_1 - \frac{2}{3}X_2\right)\right], \quad \mu = \cosh\lambda, \quad \nu = \sinh\lambda, \end{aligned}$$

$$((\Delta F)^2 = \langle 000|F^2|000\rangle - (\langle 000|F|000\rangle)^2.$$

It indicates that the uncertainty of the first quadrature U_1 in the state $|\psi\rangle_\beta$ is the same as that in the usual three-mode squeezed vacuum, while the uncertainty of the second quadrature U_2 is substantially modified with respect to the linear case.

If we choose $F_1(X_1) = X_1^2$, $F_2(X_2) = 0$, $F_3(X_3) = 0$, we can have:

$$(\Delta F)^2 = 3\pi^{\frac{3}{2}}\tau_1^2 \frac{6\cosh^2\lambda + \sinh 2\lambda - 3}{(3\cosh\lambda - \sinh\lambda)^2}. \tag{41}$$

So by the means of (42), we obtain

$$(\Delta U_2)^2 = \frac{1}{4}\exp(-2\lambda) + \frac{3}{4}\pi^{\frac{3}{2}}\tau_1^2 \frac{6\cosh^2\lambda + \sinh 2\lambda - 3}{(3\cosh\lambda - \sinh\lambda)^2}. \tag{42}$$

For $(\Delta F)^2 > 0$, we have the uncertainty of the second quadrature U_2 is more than it in the usual three-mode squeezed vacuum.

Summary In summary, we have introduced the coordinate-dependent three-mode squeezing transformations and discussed the properties of the corresponding squeezed states. We have also shown that the nonlinear three-mode coordinate-dependent Bogoliubov transformations can be constructed by the combination of two unitary transformations, a coordinate-dependent displacement followed by the standard squeezed transformation. Such decomposition has shown that the transformations are canonical under very broad constraints, i.e., $F_1 = F_1^\dagger$, $F_2 = F_2^\dagger$, $F_3 = F_3^\dagger$, $k_1 = l_1$, $\tau_2 = l_2$, $\tau_3 = k_3$, $k_2 = \frac{\tau_2(3\cosh\lambda - \sinh\lambda)}{2\sinh\lambda}$, $\tau_1 = \frac{k_1(3\cosh\lambda - \sinh\lambda)}{2\sinh\lambda}$, $l_3 = \frac{\tau_3(\cosh\lambda - \sinh\lambda)}{2\sinh\lambda}$. Such decomposition turns a nonlinear problem into an essentially linear one because all the calculations involving the nonlinear three-mode squeezed transformation have been shown to be able to reduce to those only concerning the standard three-mode squeezed states which only involves the linear and quadrature-independent Bogoliubov transformations.

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